

SKEW MONOIDAL MONOIDS

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ABSTRACT. Skew monoidal categories are monoidal categories with non-invertible ‘coherence’ morphisms. As shown in a previous paper bialgebroids over a ring R can be characterized as the closed skew monoidal structures on the category $\mathbf{Mod}\text{-}R$ in which the unit object is R_R . This offers a new approach to bialgebroids and Hopf algebroids. Little is known about skew monoidal structures on general categories. In the present paper we study the one-object case: skew monoidal monoids (SMM). We show that they possess a dual pair of bialgebroids describing the symmetries of the (co)module categories of the SMM. These bialgebroids are submonoids of their own base and are rank 1 free over the base on the source side. We give various equivalent definitions of SMM, study the structure of their (co)module categories and discuss the possible closed and Hopf structures on a SMM.

1. THREE DEFINITIONS OF SKEW MONOIDAL MONOIDS

Skew monoidal monoids are skew monoidal categories, as defined in [4], in which the underlying category has only one object, so it is a monoid.

If A denotes the monoid and 1 denotes its unit element then a skew-monoidal structure on A amounts to have a monoid morphism $A \times A \rightarrow A$, $\langle a, b \rangle \mapsto a * b$, the skew monoidal product,

$$(1) \quad (ab) * (cd) = (a * c)(b * d)$$

$$(2) \quad 1 * 1 = 1$$

and elements $\gamma, \eta, \varepsilon$ of A , the “coherence” morphisms, satisfying the naturality conditions

$$(3) \quad \gamma(a * (b * c)) = ((a * b) * c)\gamma$$

$$(4) \quad \eta a = (1 * a)\eta$$

$$(5) \quad a\varepsilon = \varepsilon(a * 1)$$

and the skew monoidality axioms

$$(6) \quad (\gamma * 1)\gamma(1 * \gamma) = \gamma^2$$

$$(7) \quad \gamma\eta = \eta * 1$$

$$(8) \quad \varepsilon\gamma = 1 * \varepsilon$$

$$(9) \quad (\varepsilon * 1)\gamma(1 * \eta) = 1$$

$$(10) \quad \varepsilon\eta = 1.$$

If the skew monoidal product were associative, so γ were the identity, and it had a unit then the Eckman-Hilton argument would force $a * b = ab$ and A would be commutative. Since γ is not assumed even to be invertible, skew monoidal monoids have a good chance to be non-trivial.

Lemma 1.1. *A skew monoidal monoid $\mathcal{A} = \langle A, *, \gamma, \eta, \varepsilon \rangle$ is the same as the data consisting of*

- (i) *a monoid A ,*

(ii) two monoid endomorphisms $T : A \rightarrow A$ and $Q : A \rightarrow A$ such that

$$(11) \quad T(a)Q(b) = Q(b)T(a) \quad a, b \in A$$

(iii) an element $\gamma \in A$ satisfying the intertwiner relations

$$(12) \quad \gamma T^2(a) = T(a)\gamma$$

$$(13) \quad \gamma TQ(a) = QT(a)\gamma$$

$$(14) \quad \gamma Q(a) = Q^2(a)\gamma$$

for all $a \in A$

(iv) and elements $\eta, \varepsilon \in A$ satisfying

$$(15) \quad \eta a = T(a)\eta$$

$$(16) \quad a\varepsilon = \varepsilon Q(a)$$

for all $a \in A$.

These data are required, furthermore, to obey the skew monoidality relations

$$(17) \quad Q(\gamma)\gamma T(\gamma) = \gamma^2$$

$$(18) \quad \gamma\eta = Q(\eta)$$

$$(19) \quad \varepsilon\gamma = T(\varepsilon)$$

$$(20) \quad Q(\varepsilon)\gamma T(\eta) = 1$$

$$(21) \quad \varepsilon\eta = 1.$$

Proof. Assume $\langle A, *, \gamma, \eta, \varepsilon \rangle$ is a SMM. Define $T := 1 * _$ and $Q := _ * 1$. Then T and Q are monoid endomorphisms by (1), (2) and satisfies (11) by the interchange law, the latter being a consequence of (1), too. Replacing two of the a, b, c by 1 in the expression $a * (b * c)$ we obtain $T^2(c)$, $TQ(b)$ and $Q(a)$, respectively. In a similar way $(a * b) * c$ leads to $T(c)$, $QT(b)$ and $Q^2(a)$. Therefore the naturality condition (3) alone implies (12), (13) and (14). Equations (15) and (16) are obvious transcriptions of (4) and (5). The same can be said about the remaining 5 skew-monoidality relations.

Vice versa, assume that the data $\langle A, T, Q, \gamma, \eta, \varepsilon \rangle$ satisfies the 11 axioms (11-21). Then a skew monoidal product on the one-object category A can be defined by $a * b := Q(a)T(b)$. The verification of the skew monoidal monoid axioms is now easy. \square

Let **Mon** denote the 2-category of monoids as a monoidal sub-2-category of the cartesian monoidal 2-category **Cat**. So the objects of **Mon** are the monoids, the 1-cells are the monoid homomorphisms $f : A \rightarrow B$ and the 2-cells $b : f \rightarrow g : A \rightarrow B$ are the elements $b \in B$ satisfying the intertwiner property $bf(a) = g(a)b$ for all $a \in A$.

As in every skew monoidal category (see [4, Lemma 2.6]) we can introduce a monad $\langle T, \mu, \eta \rangle$ and a comonad $\langle Q, \delta, \varepsilon \rangle$ on the object A of **Mon** where multiplication μ and the comultiplication δ are defined by

$$(22) \quad \mu := Q(\varepsilon)\gamma$$

$$(23) \quad \delta := \gamma T(\eta).$$

It has also been shown in that Lemma that $\gamma : TQ \rightarrow QT$ is a mixed distributive law. In our one-object category the defining relations of the distributive law take the form

$$(24) \quad Q(\mu)\gamma T(\gamma) = \gamma\mu$$

$$(25) \quad Q(\gamma)\gamma T(\delta) = \delta\gamma$$

$$(26) \quad \gamma\eta = Q(\eta)$$

$$(27) \quad \varepsilon\gamma = T(\varepsilon).$$

The last two of these coincide with the triangle relations (18-19), the first two are simple consequences of the pentagon relation (17).

The SMM axioms can be reformulated using μ and δ instead of γ . Although the number of axioms gets larger, it reveals SMM-s as something like a bimonoid.

Proposition 1.2. *A skew monoidal monoid is the same as the data consisting of*

- *a monoid A ,*
- *two monoid endomorphisms $T : A \rightarrow A$ and $Q : A \rightarrow A$*
- *and elements $\mu, \eta, \delta, \varepsilon$ of A*

subject to the following axioms.

Intertwiner relations: For all $a \in A$

$$(28) \quad \mu T^2(a) = T(a)\mu$$

$$(29) \quad \eta a = T(a)\eta$$

$$(30) \quad \delta Q(a) = Q^2(a)\delta$$

$$(31) \quad a\varepsilon = \varepsilon Q(a)$$

$$(32) \quad \mu\delta TQ(a) = QT(a)\mu\delta$$

Commutation relations: For all $a, b \in A$

$$(33) \quad T(a)Q(b) = Q(b)T(a)$$

$$(34) \quad \mu Q(a) = Q(a)\mu$$

$$(35) \quad \delta T(a) = T(a)\delta$$

Bimonoid(-like) relations:

$$(36) \quad \mu T(\mu) = \mu^2$$

$$(37) \quad \mu\eta = 1$$

$$(38) \quad \mu T(\eta) = 1$$

$$(39) \quad Q(\delta)\delta = \delta^2$$

$$(40) \quad \varepsilon\delta = 1$$

$$(41) \quad Q(\varepsilon)\delta = 1$$

$$(42) \quad Q(\mu)\mu\delta T(\delta) = \delta\mu$$

$$(43) \quad \varepsilon\mu = \varepsilon T(\varepsilon)$$

$$(44) \quad \delta\eta = Q(\eta)\eta$$

$$(45) \quad \varepsilon\eta = 1$$

Proof. Setting $\gamma := \mu\delta$ the (28-45) axioms imply the axioms of Lemma 1.1. Assuming the latter axioms and defining μ and δ by (22) and (23) the axioms (28-45) are easily obtained. \square

Every SMM A contains distinguished submonoids

$$(46) \quad Q(A) \subset F \supset S(A) \subset G \supset T(A)$$

where

$$(47) \quad F = Q(A)\delta$$

$$(48) \quad G = \mu T(A)$$

and S is the anti endomorphism

$$(49) \quad S : A \rightarrow A^{\text{op}}, \quad S(a) = \mu T(Q(a)\eta) \equiv Q(\varepsilon T(a))\delta.$$

Furthermore,

- (i) the endomorphisms Q, S, T are all injective,
- (ii) $\pi : A \rightarrow A$, $\pi(a) := \varepsilon a\eta$ is a common left inverse in **Set** of the endofunctions Q, S and T ,
- (iii) their images, $Q(A), S(A)$ and $T(A)$, pairwise commute,
- (iv) $S(A) = F \cap G$.

For example, (iv) can be shown as follows. By the two equivalent formulas in (49) it is clear that $S(A) \subset F \cap G$. Vice versa, suppose $x \in F \cap G$. Then there exist $a, b \in A$ such that $x = Q(a)\delta = \mu T(b)$. Multiplying with ε , resp. η we obtain $a = \varepsilon x$ and $b = x\eta$. Introducing $y := a\eta$ we can write $Q(y)\eta = Q(a)Q(\eta)\eta = Q(a)\delta\eta = x\eta = b$ and therefore $S(y) = \mu T(Q(y)\eta) = \mu T(b) = x$. This proves $F \cap G \subset S(A)$.

The following useful relations will also be used in the sequel without explicit mention:

$$(50) \quad \mu S(a) = \mu TQ(a)$$

$$(51) \quad S(a)\mu = \mu TS(a)$$

$$(52) \quad S(a)\delta = QT(a)\delta$$

$$(53) \quad \delta S(a) = QS(a)\delta$$

$$(54) \quad S(a)\eta = Q(a)\eta$$

$$(55) \quad \varepsilon S(a) = \varepsilon T(a).$$

2. THE CATEGORY OF MODULES

The category **Mod- \mathcal{A}** of modules over a skew monoidal monoid $\mathcal{A} = \langle A, *, \gamma, \eta, \varepsilon \rangle$ is by definition [4] the Eilenberg-Moore category \mathcal{A}^T of the canonical monad T . Since \mathcal{A}^T is just a category while the category of modules over a bialgebroid is always monoidal, the challenge is to find a monoidal structure on \mathcal{A}^T . This problem has been solved for skew monoidal categories [5], [2, Theorem 8.1] provided the underlying category has reflexive coequalizers and the skew monoidal product preserves such coequalizers in the 2nd argument. As we shall see, in case of skew monoidal monoids these assumptions are not necessary.

Since \mathcal{A} has only one object, the objects of \mathcal{A}^T are just elements $x \in A$ satisfying two equations

$$(56) \quad x\mu = xT(x)$$

$$(57) \quad x\eta = 1_A.$$

The arrows $t \in \mathcal{A}^T(x, y)$ are the elements $t \in A$ such that

$$(58) \quad tx = yT(t).$$

For a skew monoidal category the monoidal product, called the horizontal tensor product and denoted by \otimes , of two T -algebras $\alpha : TM \rightarrow M$ and $\beta : TN \rightarrow N$ is defined as follows. First we

introduce the notation $\mu_{M,N} := (\varepsilon_M * N) \circ \gamma_{M,R,N}$. Then the underlying object of $M \bar{\otimes} N$ is given by the reflexive coequalizer

$$M * TN \begin{array}{c} \xrightarrow{\mu_{M,N}} \\ \xrightarrow{M*\beta} \end{array} M * N \longrightarrow M \bar{\otimes} N$$

and the T -action on $M \bar{\otimes} N$ is given by unique factorization in the diagram

$$(59) \quad \begin{array}{ccccc} R * (N * TM) & \begin{array}{c} \xrightarrow{R*\mu_{N,M}} \\ \xrightarrow{R*(N*\alpha)} \end{array} & R * (N * M) & \xrightarrow{R*\pi} & R * (N \bar{\otimes} M) \\ \downarrow (\beta * TM) \circ \gamma_{R,N,TM} & & \downarrow (\beta * M) \circ \gamma_{R,N,M} & & \downarrow \psi \\ N * TM & \begin{array}{c} \xrightarrow{\mu_{N,M}} \\ \xrightarrow{N*\alpha} \end{array} & N * M & \xrightarrow{\pi} & N \bar{\otimes} M \end{array}$$

provided T preserves reflexive coequalizers. In case of skew monoidal monoids all these diagrams simplify radically: all objects are the same and all μ -s are the same element (22) of A . Thus the horizontal tensor product of $x, y \in \mathcal{A}^T$ is the element $x \bullet y$ of A making the diagram in \mathcal{A}

$$\begin{array}{ccccc} A & \begin{array}{c} \xrightarrow{T(\mu)} \\ \xrightarrow{T^2(y)} \end{array} & A & \xrightarrow{T(y)} & A \\ \downarrow Q(x)\gamma & & \downarrow Q(x)\gamma & & \downarrow x \bullet y \\ A & \begin{array}{c} \xrightarrow{\mu} \\ \xrightarrow{T(y)} \end{array} & A & \xrightarrow{y} & A \end{array}$$

commutative. Now it is easy to check that

$$(60) \quad x \bullet y = yQ(x)\delta.$$

Note that the 2nd row is the canonical split coequalizer of the T -algebra y , hence it is preserved by T . This proves that the horizontal tensor product of any pair of objects exists in \mathcal{A}^T for any skew monoidal monoid \mathcal{A} .

It is now an easy exercise to determine the horizontal tensor product of arrows $s \in \mathcal{A}^T(x_1, y_1)$ and $t \in \mathcal{A}^T(x_2, y_2)$. It is given by the formula

$$(61) \quad s \bar{\otimes} t = y_2 Q(s) \eta t \in \mathcal{A}^T(x_1 \bullet x_2, y_1 \bullet y_2).$$

Notice that we use different notations for horizontal tensor product of objects and of arrows. This is necessary since the same element of A can be an object and an (non-identity) arrow in \mathcal{A}^T .

Proposition 2.1. *Let $\mathcal{A} = \langle A, T, Q, \gamma, \eta, \varepsilon \rangle$ be a skew monoidal monoid. Then its module category \mathcal{A}^T equipped with the horizontal tensor product is a strict monoidal category. In this category μ is the underlying object of a comonoid*

$$\varepsilon \xleftarrow{\varepsilon} \mu \xrightarrow{\delta} \mu \bullet \mu.$$

The associated representable functor is a faithful strong monoidal functor

$$\mathcal{A}^T(\mu, -) : \mathcal{A}^T \rightarrow {}_A\text{Set}_A, \quad x \mapsto xT(A)$$

to the category of A - A -bimodules ($=A$ -bisets or $(A^{\text{op}} \times A)$ -sets).

Proof. Strict associativity of \bullet and $\bar{\otimes}$ is a routine calculation. So is unitality; the unit object is ε . By (36-37) the μ is an object of \mathcal{A}^T . It has a special property that for any object $x \in \mathcal{A}^T$ the element $x \in A$ is also an arrow $x : \mu \rightarrow x$ in \mathcal{A}^T . Indeed, compare (56) with (58). It follows that the functor $\mathcal{A}^T(\mu, -)$ is faithful: If $sx = tx$ for a pair s, t of parallel arrows $x \rightarrow y$ then $s = t$ by (57). Now $\varepsilon : \mu \rightarrow \varepsilon$ is an arrow of \mathcal{A}^T and so is $\delta : \mu \rightarrow \mu \bullet \mu$ by (42). The comonoid axioms for $\langle \mu, \delta, \varepsilon \rangle$ reduce to the comonad axioms (39-41) after noticing that for any arrow $t \in \mathcal{A}^T$ we have $1_\mu \bar{\otimes} t = t$ and $t \bar{\otimes} 1_\mu = Q(t)$, as elements of A .

Before turning to the monoidal structure of the functor $\mathcal{A}^T(\mu, -)$ we need an explicit computation of the intertwiner set $\mathcal{A}^T(\mu, x)$. If $s : \mu \rightarrow x$ is an intertwiner then $s = s\mu T(\eta) = xT(s\eta) \in xT(A)$. Vice versa, if $a \in A$ then $xT(a)\mu = x\mu T^2(a) = xT(xT(a))$, so $xT(a)$ is an intertwiner $\mu \rightarrow x$. This proves the equality $\mathcal{A}^T(\mu, x) = xT(A)$, as subsets of A . In particular, $\mathcal{A}^T(\mu, \mu) = \mu T(A)$ is a monoid containing both $T(A)$ and $S(A)$ as submonoids. This allows us to define the A - A -bimodule structure on $\mathcal{A}^T(\mu, x)$ by right multiplication

$$a_1 \cdot t \cdot a_2 := tS(a_1)T(a_2), \quad t \in \mathcal{A}^T(\mu, x), \quad a_1, a_2 \in A.$$

The comonoid structure of μ induces the following monoidal structure for the functor $\mathcal{A}^T(\mu, -)$.

$$(62) \quad \begin{aligned} \mathcal{A}^T(\mu, x) \otimes_A \mathcal{A}^T(\mu, y) &\rightarrow \mathcal{A}^T(\mu, x \bullet y) \\ s \otimes t &\mapsto (s \bar{\otimes} t) \circ \delta = tQ(s)\delta \end{aligned}$$

and the arrow $A \rightarrow \mathcal{A}^T(\mu, \varepsilon)$ in ${}_A\text{Set}_A$ is given by $1_A \mapsto \varepsilon$ which is well-defined since $a \cdot \varepsilon = \varepsilon S(a) = \varepsilon T(a) = \varepsilon \cdot a$ by (55). This monoidal structure is strong: The inverse of (62) is

$$r \in \mathcal{A}^T(\mu, x \bullet y) \mapsto x \otimes_A yT(r\eta) \in \mathcal{A}^T(\mu, x) \otimes_A \mathcal{A}^T(\mu, y)$$

and the inverse of the identity constraint is

$$r \in \mathcal{A}^T(\mu, \varepsilon) \mapsto r\eta \in A$$

since $\mathcal{A}^T(\mu, \varepsilon) = \varepsilon T(A)$. □

Obviously, the next task is to factorize the forgetful functor of the above Proposition through the forgetful functor of G -modules, where $G = \mathcal{A}^T(\mu, \mu)$, and show that G is a right bialgebroid over A . The conspicuous property of this bialgebroid is that it is a submonoid of its own base monoid.

Theorem 2.2. *The forgetful functor of Proposition 2.1 factorizes through the forgetful functor $\text{Set}_G \rightarrow {}_A\text{Set}_A$ of a right A -bialgebroid G via a fully faithful $\mathcal{A}^T \rightarrow \text{Set}_G$ where the bialgebroid G is defined by*

$$\begin{aligned} \text{underlying monoid} &:= \mathcal{A}^T(\mu, \mu) = \mu T(A) \\ \text{source } s^G : A &\rightarrow G, \quad s^G(a) := T(a) \\ \text{target } t^G : A^{\text{op}} &\rightarrow G, \quad t^G(a) := S(a) \\ \text{comultiplication } \Delta^G : G &\rightarrow G \otimes_A G, \quad \Delta^G(g) := \mu \otimes_A \mu T(\delta g \eta) \\ \text{counit } \epsilon^G : G &\rightarrow A, \quad \epsilon^G(g) := \pi(g) = \varepsilon g \eta. \end{aligned}$$

Proof. The verification of the right bialgebroid axioms is a short exercise. The functor $\mathcal{A}^T \rightarrow \text{Set}_G$ is faithful since the functor $\mathcal{A}^T \rightarrow {}_A\text{Set}_A$ was already faithful. In order to prove fullness let $f : xT(A) \rightarrow yT(A)$ be a G -module map, $f(sg) = f(s)g$. Then $f(x)\mu = f(x)T(x)$ and, writing $f(x) = yT(t)$ with a unique $t \in A$, we obtain $yT(t)\mu = yT(tx)$ hence $yT(t) = tx$. This proves that $t \in \mathcal{A}^T(x, y)$ and $f(s) = f(xT(s\eta)) = f(x)T(s\eta) = yT(ts\eta) = ts$ therefore the functor is full. □

As usual for right bialgebroids we consider G as an A -bimodule via $a_1 \cdot g \cdot a_2 = gS(a_1)T(a_2)$. Notice that μ is a free generator for G as right A -module, the G_A is free of rank 1. This explains the simple formula for the coproduct since $G \otimes_A G \cong G$ at least as right A -modules. The left A -module structure is by far not so trivial.

Definition 2.3. A right module M over a right A -bialgebroid G (or a left module over a left bialgebroid) is called source-regular if the A -module obtained from M by restriction along the source map $s^G : A \rightarrow G$ is isomorphic to the right regular A -module A_A (resp. the left regular ${}_A A$).

Corollary 2.4. Let \mathcal{A} be a SMM. Then there is a source regular right bialgebroid G such that $\mathbf{Mod}\text{-}\mathcal{A} \equiv \mathcal{A}^T$ is monoidally equivalent to the category $\mathbf{Set}_G^{\text{reg}}$ of source regular right G -modules.

Proof. Clearly, the image of the forgetful functor $\mathcal{A}^T(\mu, -)$ consist only of the source-regular modules $xT(A)$ for $x \in \text{ob } \mathcal{A}^T$.

Let M be a source-regular right G -module and let $m_0 \in M$ be a free generator of M_A , so $A \rightarrow M$, $a \mapsto m_0 \cdot T(a)$ is a bijection. Then $m_0 \cdot \mu = m_0 \cdot T(x)$ for a unique $x \in A$. From the equations

$$m_0 \cdot T(x\mu) = m_0 \cdot \mu T(\mu) = m_0 \cdot \mu^2 = m_0 \cdot T(x)\mu = m_0 \cdot \mu T^2(x) = m_0 \cdot T(xT(x))$$

it follows that $x\mu = xT(x)$. Similarly, $m_0 \cdot T(x\eta) = m_0 \cdot \mu T(\eta) = m_0$ implies $x\eta = 1$. This proves that x is a T -algebra such that $M \cong xT(A)$ as G -modules. \square

3. COMODULES AND THE DUAL BIALGEBROID

We shall be very brief about comodules since all what we have to say can be obtained from the structure of modules by dualizing, i.e., $\mathbf{Comod}\text{-}\mathcal{A} = \mathcal{A}_Q$ is nothing but $\mathbf{Mod}\text{-}\mathcal{A}^{\text{op, rev}}$.

The objects of \mathcal{A}_Q are elements $u \in A$ satisfying $Q(u)u = \delta u$ and $\varepsilon u = 1$. The arrows $s \in \mathcal{A}_Q(u, v)$ are elements $s \in A$ satisfying $Q(s)u = vs$.

Proposition 3.1. The Eilenberg-Moore category \mathcal{A}_Q of the comonad Q is a strict monoidal category w.r.t. the vertical tensor product defined for objects by $u \odot v := \mu T(v)u$ and for arrows $s : u_1 \rightarrow v_1$, $t : u_2 \rightarrow v_2$ by $s \otimes t := t\varepsilon T(s)u_2$. The element $\delta \in A$ is the underlying object of the monoid $\langle \delta, \mu, \eta \rangle$ in \mathcal{A}_Q and δ is a cogenerator in \mathcal{A}_Q . The endomorphism monoid of δ is the underlying monoid of a source-regular left A -bialgebroid F defined by

$$\begin{aligned} \text{underlying monoid} : \mathcal{A}_Q(\delta, \delta) &= Q(A)\delta \\ \text{source } s_F : A &\rightarrow F, \quad s_F(a) = Q(a) \\ \text{target } t_F : A^{\text{op}} &\rightarrow F, \quad t_F(a) = S(a) \\ \text{comultiplication } \Delta_F : {}_A F_A &\rightarrow {}_A (F \otimes_A F)_A, \quad \Delta_F(f) = Q(\varepsilon f \mu) \delta \otimes_A \delta \\ \text{counit } \epsilon_F : {}_A F_A &\rightarrow {}_A A_A, \quad \epsilon_F(f) = \pi(f) \equiv \varepsilon f \eta \end{aligned}$$

where in the last two lines the A -bimodule structure of F is defined by $a_1 \cdot f \cdot a_2 := Q(a_1)S(a_2)f$.

Theorem 3.2. The forgetful functor $\mathcal{A}_Q(-, \delta) : \mathcal{A}_Q^{\text{op}} \rightarrow {}_A \mathbf{Set}_A$ is strong monoidal and factors through a monoidal equivalence $\mathcal{A}_Q^{\text{op}} \cong {}_F^{\text{reg}} \mathbf{Set}$ of $\mathcal{A}_Q^{\text{op}}$ with the category of source-regular left F -modules.

In order to characterize \mathcal{A}_Q in terms of G we need to perform a duality transformation also on the level of bialgebroids.

Lemma 3.3. The left dual $G^* = \text{Hom}(G_A, A_A)$ of the right A -bialgebroid G can be identified with the left A -bialgebroid F by the non-degenerate pairing

$$\langle -, - \rangle : F \times G \rightarrow A, \quad \langle f, g \rangle := \varepsilon f g \eta$$

satisfying the following properties

$$(63) \quad \langle f, a_1 \cdot g \cdot a_2 \rangle = \langle fQ(a_1), g \rangle a_2$$

$$(64) \quad \langle a_1 \cdot f \cdot a_2, g \rangle = a_1 \langle f, T(a_2)g \rangle$$

$$(65) \quad \langle f, gg' \rangle = \langle f_{(1)} \cdot \langle f_{(2)}, g \rangle, g' \rangle$$

$$(66) \quad \langle ff', g \rangle = \langle f, \langle f', g^{(1)} \rangle \cdot g^{(2)} \rangle$$

$$(67) \quad \langle f, 1_G \rangle = \pi(f)$$

$$(68) \quad \langle 1_F, g \rangle = \pi(g).$$

Corollary 3.4. *The category \mathcal{A}_Q of comodules of the SMM \mathcal{A} is monoidally equivalent to the category $\mathbf{Set}_{\text{reg}}^G$ of source-regular right G -comodules.*

Proof. By dualizing Corollary 2.4 every source-regular left F -module $X \in {}_F^{\text{reg}}\mathbf{Set}$ is isomorphic to $Q(A)u$ for some $u \in \text{ob } \mathcal{A}_Q$. The right dual ${}^*X = \text{Hom}({}_A X, {}_A A)$ is therefore isomorphic to

$${}^*[Q(A)u] = \text{Hom}(Q(A)u, {}_A A) = \{\varphi : Q(A)u \rightarrow A \mid \exists a_0 \in A, \varphi(x) = \varepsilon x a_0\}$$

and left F -module structures on X and right G -comodule structures on *X are in bijection via

$$\langle f \cdot x, \varphi \rangle = \langle f, \langle x, \varphi^{(0)} \rangle \cdot \varphi^{(1)} \rangle, \quad f \in F, x \in X, \varphi \in {}^*X.$$

For source-regular F -modules X and Y and for a left A -module map $h : X \rightarrow Y$ its transpose ${}^*h : {}^*Y \rightarrow {}^*X$ is a right A -module map and

$$\begin{aligned} \langle f \cdot h(x), \varphi \rangle &= \langle f, \langle x, {}^*h(\varphi^{(0)}) \rangle \cdot \varphi^{(1)} \rangle \\ \langle h(f \cdot x), \varphi \rangle &= \langle f, \langle x, {}^*h(\varphi)^{(0)} \rangle \cdot {}^*h(\varphi)^{(1)} \rangle \end{aligned}$$

Therefore h is left F -linear precisely when *h is right G -colinear. \square

Together with F and G the monoid A contains also an image of the smash product $G \# F$. Indeed, the axiom (42) implies that for all $g \in G, f \in F$

$$\begin{aligned} fg &= Q(\varepsilon f) \delta \mu T(g\eta) = Q(\varepsilon f \mu) \mu \delta T(\delta g\eta) = \mu T(\delta g\eta) Q(\varepsilon f \mu) \delta = \\ &= g^{(2)} S(\langle f_{(2)}, g^{(1)} \rangle) f_{(1)} = (f_{(2)} \rightharpoonup g) f_{(1)} \end{aligned}$$

where $f \rightharpoonup g$ is the action that makes G a left module algebra over the cooposite (left bialgebroid) of F . The invariant submonoid is

$$G^F = \{g \in G \mid f \rightharpoonup g = \pi(f) \cdot g\} = T(A).$$

The smash product $G \# F = G \otimes_{S(A)} F$ is the tensor product over the common submonoid $S(A) = F \cap G$ with multiplication rule

$$(g \# f)(g' \# f') = g(f_{(2)} \rightharpoonup g') \# f_{(1)} f' = gg'^{(2)} S(\langle f_{(2)}, g'^{(1)} \rangle) \# f_{(1)} f'.$$

There is a remarkable connection between grouplike elements of G and the objects of \mathcal{A}_Q . Recall that in a bialgebroid G an element $g \in G$ is called grouplike if $\Delta^G(g) = g \otimes_A g$ and $\epsilon^G(g) = 1$.

Lemma 3.5. *The bijection $A \xrightarrow{\sim} G, a \mapsto \mu T(a)$, restricts to a bijection $A_Q \xrightarrow{\sim} \mathbf{Gr}(G)$ between the set of objects $u \in \mathcal{A}_Q$ and the set of grouplike elements of the bialgebroid G . This bijection lifts to an isomorphism of monoids: $A_Q \xrightarrow{\sim} \mathbf{Gr}(G)^{\text{op}}$. Dually, the map $x \mapsto Q(x)\delta$ defines an isomorphism of monoids $A^T \xrightarrow{\sim} \mathbf{Gr}(F)^{\text{op}}$ from the monoid of objects of \mathcal{A}^T to the opposite of the monoid of grouplikes in F .*

We do not know other objects than the tensor powers of μ in \mathcal{A}^T and those of δ in \mathcal{A}_Q . The known part of \mathcal{A}^T , for example, contains the simplicial object

$$\begin{array}{ccccccc}
 & & \xleftarrow{Q(\varepsilon)} & & \xleftarrow{Q^2(\varepsilon)} & & \\
 & & \xleftarrow{\delta} & & \xleftarrow{Q(\delta)} & & \\
 \varepsilon & \xleftarrow{\varepsilon} & \mu & \xrightarrow{\delta} & \mu \bullet \mu & \xleftarrow{Q(\varepsilon)} & \mu \bullet \mu \bullet \mu \quad \cdots \\
 & & \xleftarrow{\varepsilon} & & \xleftarrow{\delta} & & \\
 & & & & \xleftarrow{\varepsilon} & &
 \end{array}$$

associated to the comonad $-\bar{\otimes} 1_\mu$ but it contains also endoarrows $g \in G = \mathcal{A}^T(\mu, \mu)$ at all occurrence of the object μ . E.g., there are arrows $\mu : \mu \rightarrow \mu$, $\mu \bar{\otimes} 1_\mu : \mu \bullet \mu \rightarrow \mu \bullet \mu$, \dots which do not belong to the above simplicial object. There is also another simplicial object $\Delta^{\text{op}} \rightarrow \mathcal{A}^T$ associated to the comonad $1_\mu \bar{\otimes} -$. Dually, the comodule category \mathcal{A}_Q contains cosimplicial objects and endoarrows $f \in F$ of the cogenerator object δ . The conjecture is that the category generated by these objects and arrows exhausts all of \mathcal{A}^T and \mathcal{A}_Q , respectively, when \mathcal{A} is the free SMM.

4. SKEW MONOIDAL MONOIDS ARE SOURCE REGULAR BIALGEBROIDS

In the previous sections we have seen that every SMM \mathcal{A} determines a dual pair of source regular bialgebroids F and G . In this section we show that every source regular bialgebroid determines a skew monoidal structure on its base monoid and these two constructions are inverses of each other, up to isomorphisms.

First we need a precise notion of isomorphism of SMM-s. Consider a skew monoidal functor $\langle A, T, Q, \gamma, \eta, \varepsilon \rangle \xrightarrow{\Phi} \langle A', T', Q', \gamma', \eta', \varepsilon' \rangle$. Such a functor consists of a monoid morphism $\phi : A \rightarrow A'$ and elements $\phi_2, \phi_0 \in A'$ satisfying the intertwiner relations (expressing naturality of ϕ_2)

$$(69) \quad \phi_2 Q' \phi(a) = \phi Q(a) \phi_2 \quad \text{and} \quad \phi_2 T' \phi(a) = \phi T(a) \phi_2$$

and the identities (being the 3 skew monoidal functor axioms)

$$(70) \quad \phi(\gamma) \phi_2 T'(\phi_2) = \phi_2 Q'(\phi_2) \gamma'$$

$$(71) \quad \phi_2 Q'(\phi_0) \eta' = \phi(\eta)$$

$$(72) \quad \phi(\varepsilon) \phi_2 T'(\phi_0) = \varepsilon'$$

Definition 4.1. An isomorphism $\Phi : \mathcal{A} \xrightarrow{\sim} \mathcal{A}'$ of skew monoidal monoids is a skew monoidal functor $\langle \phi, \phi_2, \phi_0 \rangle : \mathcal{A} \rightarrow \mathcal{A}'$ such that $\phi : A \rightarrow A'$ is an isomorphism of monoids and the elements ϕ_2 and ϕ_0 of A' are invertible.

In particular, an isomorphism of SMM-s is both a skew monoidal and a skew opmonoidal functor. Therefore it determines not only a monad morphism

$$(73) \quad \sigma := \phi_2 Q'(\phi_0) : T' \phi \rightarrow \phi T$$

$$(74) \quad \sigma \mu' = \phi(\mu) \sigma T'(\sigma)$$

$$(75) \quad \sigma \eta' = \phi(\eta)$$

but a comonad morphism

$$(76) \quad \tau := T'(\phi_0^{-1})\phi_2^{-1} : \phi Q \rightarrow Q'\phi$$

$$(77) \quad \delta'\tau = Q'(\tau)\tau\phi(\delta)$$

$$(78) \quad \varepsilon'\tau = \phi(\varepsilon)$$

as well.

Proposition 4.2. *Let \mathcal{A} and \mathcal{A}' be skew monoidal monoids and let G , resp. G' , denote the right A -bialgebroid associated to \mathcal{A} by Theorem 2.2 and the right A' -bialgebroid associated to \mathcal{A}' . If $\Phi : \mathcal{A} \rightarrow \mathcal{A}'$ is an isomorphism of skew monoidal monoids then the pair $\langle \varphi, \varphi_0 \rangle$, where*

$$\begin{aligned} \varphi : G &\rightarrow G' & g &\mapsto \tau\phi(g)\tau^{-1} \\ \varphi_0 : A &\rightarrow A', & a &\mapsto \phi_0^{-1}\phi(a)\phi_0 \end{aligned}$$

is an isomorphism of bialgebroids from G to G' , i.e., the following identities hold for all $a \in A$ and $g \in G$:

$$(79) \quad \varphi(T(a)) = T'(\varphi_0(a))$$

$$(80) \quad \varphi(S(a)) = S'(\varphi_0(a))$$

$$(81) \quad \epsilon^{G'}(\varphi(g)) = \varphi_0(\epsilon^G(g))$$

$$(82) \quad \varphi(g)^{(1)} \otimes_{A'} \varphi(g)^{(2)} = \varphi(g^{(1)}) \otimes_{A'} \varphi(g^{(2)}).$$

Proof. Notice that for $g \in G$ the expression $\sigma^{-1}\phi(g)\sigma$ is a well-defined composition of arrows $\mu' \rightarrow \phi(\mu)\sigma \rightarrow \phi(\mu)\sigma \rightarrow \mu'$ in $\mathcal{A}'^{T'}$, therefore defines an element in G' . But φ was defined using τ instead of σ^{-1} . Fortunately, the difference is only an inner automorphism of G' , induced by $T'(\phi_0)$, so $\varphi : G \rightarrow G'$ is well-defined.

The verification of the relations (79 - 82) requires too much place to account for in full detail. But it is straightforward using the (co)monad morphism relations and, occasionally, the fact that since ϕ is an isomorphism, expressions like $\phi_2 Q'(\phi_0)\phi_2^{-1}$ commute with $\phi(g)$ for $g \in G$. \square

Theorem 4.3. *Let A be a monoid. Then there is a bijection between isomorphism classes of skew monoidal monoids $\langle A, T, Q, \mu, \eta, \delta, \varepsilon \rangle$ with underlying monoid A and source regular right bialgebroids $\langle G, A, s^G, t^G, \Delta^G, \epsilon^G \rangle$ over A .*

Proof. The construction of G from a SMM \mathcal{A} given in Theorem 2.2 serves as the object map of a functor from the category of SMM-s, with arrows being the isomorphisms of Definition 4.1, into the category of source regular bialgebroids. The arrow map is provided by Proposition 4.2.

Next we construct a map from source regular bialgebroids to SMM-s. Source regularity is precisely the statement that the source map $s^G : A \rightarrow G$, as a 1-cell in \mathbf{Mon} , has a right adjoint. We choose a right adjoint $s_G : G \rightarrow A$ with unit $\eta : \text{id}_A \rightarrow s_G s^G$ and counit $\mu : s^G s_G \rightarrow \text{id}_G$. Then both s_G and s^G are injective and $a \mapsto \mu s^G(a)$ is a bijection $A \rightarrow G$ with inverse $g \mapsto s_G(g)\eta$. From the adjunction relations it follows that $T := s_G s^G$ is a monad on A with multiplication $s_G(\mu)$ and unit η . This proves 5 of the 18 SMM axioms of Proposition 1.2, namely (28), (29), (36), (37) and (38).

Let $F := \text{Hom}(G_A, A_A)$ with $\langle f, g \rangle$ denoting evaluation of $f \in F$ on $g \in G$. Then F is a left A -module via $\langle a \cdot f, g \rangle = a \langle f, g \rangle$ and every $f \in F$ is uniquely determined by its value on μ since $\langle f, \mu s^G(a) \rangle = \langle f, \mu \rangle a$. This defines a bijection $J : F \rightarrow A$, $J(f) := \langle f, \mu \rangle$. Using the comonoid structure of G we make F into a monoid by (66) and (68). Then defining

$$(83) \quad \varepsilon := J(1_F), \quad \delta := J^{-1}(1_A), \quad s_F(a) := J^{-1}(a\varepsilon), \quad s^F(f) := J(\delta f)$$

we obtain that $J(s_F(a)) = a\varepsilon = J(a \cdot 1_F)$, hence $s_F(a) = a \cdot 1_F$ and

$$\begin{aligned} \langle f s_F(a), g \rangle &= \langle f, \langle s_F(a), g^{(1)} \rangle \cdot g^{(2)} \rangle = \langle f, a \cdot g \rangle = \\ (84) \quad &= \langle f, g t^G(a) \rangle. \end{aligned}$$

We deduce that $s_F : A \rightarrow F$ and $s^F : F \rightarrow A$ are monoid morphisms and s^F is left adjoint to the source map s_F with unit δ and counit ε . It follows that $Q := s^F s_F$ is a comonad on A with comultiplication $s^F(\delta)$ and counit ε . This proves 5 more SMM axioms.

The canonical pairing can now be written as

$$(85) \quad \langle f, g \rangle = \varepsilon s^F(f) s_G(g) \eta.$$

The target map of the dual (would-be-)bialgebroid F can be introduced by

$$(86) \quad \langle t_F(a), g \rangle := \langle 1_F, s^G(a)g \rangle$$

and proving

$$\begin{aligned} \langle f t_F(a), g \rangle &= \langle f, \langle t_F(a), g^{(1)} \rangle \cdot g^{(2)} \rangle = \langle f, \varepsilon^G(s^G(a)g^{(1)}) \cdot g^{(2)} \rangle \\ &= \langle f, \varepsilon^G(g^{(1)}) \cdot t^G(a)g^{(2)} \rangle = \\ (87) \quad &= \langle f, t^G(a)g \rangle \end{aligned}$$

it follows that it is a monoid morphism $t_F : A^{\text{op}} \rightarrow F$ and $t_F(a)$ commutes with $s_F(b)$ for all $a, b \in A$. What used to be the antiendomorphism S in previous Sections can now be written as

$$(88) \quad s^F t_F(a) = \langle \delta t_F(a), \mu \rangle = \langle \delta, t^G(a)\mu \rangle = s_G t^G(a) \eta.$$

There is one more property of the pairing that we need:

$$\begin{aligned} \langle t_F(a)f, g \rangle &= \langle t_F(a), \langle f, g^{(1)} \rangle \cdot g^{(2)} \rangle \stackrel{(86)}{=} \langle 1_F, \langle f, g^{(1)} \rangle \cdot s^G(a)g^{(2)} \rangle = \\ &= \langle 1_F, \langle f, (s^G(a)g)^{(1)} \rangle \cdot (s^G(a)g)^{(2)} \rangle = \\ (89) \quad &= \langle f, s^G(a), g \rangle. \end{aligned}$$

The basic commutation relations of a SMM structure can now be proven as follows. Since

$$s^F s_F(a) = \langle \delta s_F(a), \mu \rangle \stackrel{(84)}{=} \langle \delta, \mu t^G(a) \rangle \stackrel{(85)}{=} s_G(\mu) s_G t^G(a) \eta,$$

we can conclude that

$$\begin{aligned} s^F s_F(a) s_G(g) &= s_G(\mu) s_G t^G(a) s_G s^G s_G(g) \eta = s_G(\mu) s_G s^G s_G(g) s_G t^G(a) \eta = \\ (90) \quad &= s_G(g) s^F s_F(a). \end{aligned}$$

Similarly, from the equality

$$s_G s^G(a) = \langle \delta, s^G(a)\mu \rangle \stackrel{(89)}{=} \langle t_F(a)\delta, \mu \rangle = \varepsilon s^F t_F(a) s^F(\delta)$$

we infer

$$\begin{aligned} s^F(f) s_G s^G(a) &= \varepsilon s^F s_F s^F(f) s^F t_F(a) s^F(\delta) = \varepsilon s^F t_F(a) s^F s_F s^F(f) s^F(\delta) = \\ (91) \quad &= s_G s^G(a) s^F(f). \end{aligned}$$

Equations (90) and (91) prove, redundantly, the 3 SMM axioms (33), (34) and (35).

It remains to show the intertwiner relation (32) for $\gamma = s_G(\mu)s^F(\delta)$ and the last 4 bimonoid axioms. For that first one proves the relations analogous to (50), (52), (54) and (55), namely

$$(92) \quad s^F t_F(a)\eta = s^F s_F(a)\eta$$

$$(93) \quad \mu t^G(a) = \mu s^G s^F s_F(a)$$

$$(94) \quad \varepsilon s_G t^G(a) = \varepsilon s_G s^G(a)$$

$$(95) \quad t_F(a)\delta = s_F s_G s^G(a)\delta.$$

Then

$$\begin{aligned} & s_G(\mu)s^F(\delta) s_G s^G s^F s_F(a) \stackrel{(91)}{=} s_G(\mu)s_G s^G s^F s_F(a)s^F(\delta) \stackrel{(93)}{=} \\ & = s_G(\mu)s_G t^G(a)s^F(\delta) \stackrel{(88)}{=} s_G(\mu)s^F t_F(a)s^F(\delta) \stackrel{(95)}{=} s_G(\mu)s^F s_F s_G s^G(a)s^F(\delta) \stackrel{(90)}{=} \\ & = s^F s_F s_G s^G(a)s_G(\mu)s^F(\delta) \end{aligned}$$

which is precisely the intertwiner relation (32). Finally, the remaining bimonoid relations can be shown by the calculations

$$\begin{aligned} & \varepsilon \eta \stackrel{(85)}{=} \langle 1_F, 1_G \rangle = \varepsilon^G(1_G) = 1_A \\ & \varepsilon s_G(\mu) \stackrel{(85)}{=} \langle 1_F, \mu^2 \rangle = \varepsilon^G(\mu^2) = \varepsilon^G(s^G(\varepsilon^G(\mu))\mu) \stackrel{(85)}{=} \langle 1_F, s^G(\varepsilon)\mu \rangle = \\ & \stackrel{(89)}{=} \langle t_F(\varepsilon), \mu \rangle \stackrel{(85)}{=} \varepsilon s^F t_F(\varepsilon) \stackrel{(88)}{=} \varepsilon s_G t^G(\varepsilon) \stackrel{(94)}{=} \varepsilon s_G s^G(\varepsilon) \\ & s^F(\delta)\eta \stackrel{(85)}{=} \langle \delta^2, 1_G \rangle = \langle \delta, \langle \delta, 1_G \rangle \cdot 1_G \rangle \stackrel{(85)}{=} \langle \delta, t^G(\eta) \rangle = \\ & \stackrel{(85)}{=} s_G t^G(\eta)\eta \stackrel{(88)}{=} s^F t_F(\eta)\eta \stackrel{(92)}{=} s^F s_F(\eta)\eta \\ & s^F(\delta)s_G(\mu) \stackrel{(85)}{=} \langle \delta^2, \mu^2 \rangle = \langle \delta, \langle \delta, \mu^{(1)}\mu^{(1')} \rangle \cdot \mu^{(2)}\mu^{(2')} \rangle = \\ & = \langle \delta, \langle \delta, \mu^2 \rangle \cdot (\mu s^G s^F(\delta))^2 \rangle = \langle \delta, s_G(\mu) \cdot \mu^2 s^G s_G s^G s^F(\delta) s^G s^F(\delta) \rangle = \\ & = \langle \delta, s_G(\mu) \cdot \mu^2 \rangle s_G s^G s^F(\delta) s^F(\delta) = \langle \delta, \mu^2 t^G s_G(\mu) \rangle s_G s^G s^F(\delta) s^F(\delta) = \\ & \stackrel{(93)}{=} \langle \delta, \mu^2 s^G s^F s_F s_G(\mu) \rangle s_G s^G s^F(\delta) s^F(\delta) = \\ & = s_G(\mu)s^F s_F s_G(\mu)s_G s^G s^F(\delta)s^F(\delta) \end{aligned}$$

where for the last relation, in passing from the first line to the second, we used the coproduct formula $\Delta^G(\mu) = \mu \otimes_A \mu T(\delta)$ which, in the present construction, can be easily proven by pairing it with arbitrary $f, f' \in F$. This finishes the construction of a skew monoidal monoid

$$\mathcal{A} = \langle A, s_G s^G, s^F s_F, s_G(\mu), \eta, s^F(\delta), \varepsilon \rangle$$

from the data of a source regular right A -bialgebroid G and from a choice of adjunction data s_G, μ, η for the left adjoint s^G .

If G is obtained from a SMM structure on A , let us say \mathcal{A}' , then s^G has another right adjoint, the inclusion $s'_G : G \hookrightarrow A$ with counit μ' and unit η' . Then there is an invertible 2-cell $z : s_G \rightarrow s'_G$ in \mathbf{Mon} such that $\mu' s^G(z) = \mu$ and $z\eta = \eta'$. Then (83) yields $\varepsilon' = \varepsilon z^{-1}$, $\delta' = s_F(z)\delta$, $s'_F(a) = s_F(a)$ and $s'^F(f) = z s^F(f) z^{-1}$. It follows that the monoid automorphism $\phi(a) = z a z^{-1}$ together with

$$\phi_2 = Q'(z^{-1})T'(z^{-1}) \quad \text{and} \quad \phi_0 = z$$

is an isomorphism of skew monoidal monoids from \mathcal{A} to \mathcal{A}' , that is to say the triple $\langle \phi, \phi_2, \phi_0 \rangle$ satisfies equations (69), (70), (71) and (72).

Going in the opposite direction suppose that we start from a source regular bialgebroid G , perform the above construction of a SMM \mathcal{A} and then apply Theorem 2.2 to construct a bialgebroid G' . This bialgebroid G' is nothing but the submonoid of A generated by $s_G(\mu)$ and $s_G s^G(A)$. But this is nothing but the image of the map s_G . Therefore restricting s_G to its image (together with the identity $A \rightarrow A$) is a bialgebroid isomorphism $G \xrightarrow{\sim} G'$. \square

5. CLOSED AND HOPF SKEW MONOIDAL MONOIDS

Let \mathcal{M} be the category of all (small) right A -modules. Then the A -bialgebroid G of a SMM \mathcal{A} determines a closed skew monoidal structure on \mathcal{M} with skew monoidal product

$$(96) \quad M \circledast N := M \underset{s}{\otimes} (N \underset{T}{\otimes} G)$$

where $\underset{s}{\otimes}$ refers to tensoring over A w.r.t the left A -action $a \cdot g = gS(a)$ on G and $\underset{T}{\otimes}$ w.r.t. the other left A -action $a \cdot g = T(a)g$. That is to say, $M \circledast N$ is the right A -set the elements of which are equivalence classes $[m, n, g]$ of elements $\langle m, n, g \rangle \in M \times N \times G$ w.r.t. the equivalence relation generated by

$$\langle m \cdot b, n \cdot a, g \rangle \sim \langle m, n, T(a)gS(b) \rangle \quad m \in M, n \in N, g \in G, a, b \in A.$$

The A -action is given by $[m, n, g] \cdot a := [m, n, gT(a)]$. The skew unit object R is the right regular A -module and the coherence morphisms are

$$\begin{aligned} \gamma_{L, M, N} &: L \circledast (M \circledast N) \rightarrow (L \circledast M) \circledast N \\ &[l, [m, n, g], h] \mapsto [[l, m, h^{(1)}], n, gh^{(2)}] \\ \eta_M &: M \rightarrow R \circledast M \\ &m \mapsto [1_A, m, 1_G] \\ \varepsilon_M &: M \circledast R \rightarrow M \\ &[m, a, g] \mapsto m \cdot \pi(T(a)g). \end{aligned}$$

The skew monoidal product $M \circledast N$ being a colimit, both endofunctors $M \circledast _$ and $_ \circledast N$ on $\mathcal{M} = \mathbf{Set}_A$ have right adjoints therefore \circledast is a closed skew monoidal structure.

Let $\mathcal{M}_1 \subset \mathcal{M}$ be the full subcategory of rank 1 free A -modules. For each object $M \in \mathcal{M}_1$ choose a free generator $\xi_M \in M$, so that

$$\nu_M : R \rightarrow M, \quad a \mapsto \xi_M \cdot a$$

is an isomorphism in \mathcal{M}_1 . For two objects $M, N \in \mathcal{M}_1$ define

$$\nu_{M, N} : M \circledast N \rightarrow G, \quad [m, n, g] \mapsto T(\nu_N^{-1}(n))gS(\nu_M^{-1}(m))$$

which is an isomorphism in \mathcal{M} with inverse $\nu_{M, N}^{-1}(g) = [\xi_M, \xi_N, g]$. Since $G \in \mathcal{M}_1$ with $\xi_G = \mu$, the category \mathcal{M}_1 is a skew monoidal subcategory of \mathcal{M} .

Lemma 5.1. *The functor $\nabla : \mathcal{M}_1 \rightarrow \mathcal{A}$ which maps $M \xrightarrow{f} N$ to $\nu_N^{-1}(f(\xi_M))$ is a skew monoidal equivalence.*

Proof. Since all objects of \mathcal{M}_1 are isomorphic, ∇ is clearly an equivalence. The only question is strong skew monoidality of ∇ . Omitting the details of a straightforward calculation we remark that the isomorphism $\nabla(M) \circledast \nabla(N) \xrightarrow{\sim} \nabla(M \circledast N)$ is given by the unique element $\alpha_{M, N} \in A$ such that

$$\xi_{M \circledast N} \cdot \alpha_{M, N} = [\xi_M, \xi_N, \mu]$$

and choosing $\xi_R = 1_A$, hence $\nu_R = \text{id}_A$, the ∇ can be made strictly normal. \square

Corollary 5.2. *Every SMM is equivalent, as skew monoidal categories, to a full skew monoidal subcategory of a closed skew monoidal category.*

Remark 5.3. Although all objects of \mathcal{M}_1 are isomorphic to the skew monoidal unit, the components of γ , η and ε may have non-invertible components within \mathcal{M}_1 . In fact if any one of the components of η or ε is invertible then all of them are and \mathcal{A} is a trivial SMM; see Lemma 5.9.

Before discussing the closed structure of SMM-s we prove an embedding theorem relating SMM-s to SMC-s. In the following Theorem *skew monoidal embedding* means a fully faithful strong skew monoidal functor.

Theorem 5.4. *Let $\langle \mathcal{M}, \otimes, R, \gamma, \eta, \varepsilon \rangle$ be a skew monoidal category (SMC). Then for the existence of a SMM \mathcal{A} and a skew monoidal embedding*

$$E : \mathcal{A} \rightarrow \mathcal{M}$$

it is sufficient and necessary that $R \otimes R \cong R$.

Proof. Let $E : \mathcal{A} \rightarrow \mathcal{M}$ be a SM embedding of a SMM \mathcal{A} . Let R' be the image of the single object of \mathcal{A} . Then $R \cong R'$ and $R' \otimes R' \cong R'$ by strong skew monoidality of E . Hence $R \otimes R \cong R$.

Assuming that $\kappa : R \otimes R \xrightarrow{\sim} R$ is an isomorphism we can define a skew monoidal structure on the endomorphism monoid $A = \mathcal{M}(R, R)$ by setting

$$\begin{aligned} a * b &:= \kappa \circ (a \otimes b) \circ \kappa^{-1}, & a, b &\in A \\ \gamma &:= \kappa \circ (\kappa \otimes R) \circ \gamma_{R,R,R} \circ (R \otimes \kappa^{-1}) \circ \kappa^{-1} \\ \eta &:= \kappa \circ \eta_R \\ \varepsilon &:= \varepsilon_R \circ \kappa^{-1} \end{aligned}$$

Therefore $\mathcal{A} = \langle A, *, \gamma, \eta, \varepsilon \rangle$ is a skew monoidal monoid and

$$E : \mathcal{A} \rightarrow \mathcal{M}, \quad E(a) = (R \xrightarrow{a} R)$$

is a skew monoidal embedding the strong skew monoidal structure of which is given by κ and by the identity arrow $R \xrightarrow{\text{id}} R$. \square

As a skew monoidal category \mathcal{A} being (left or/and right) closed is equivalent, by Lemma 5.1, to \mathcal{M}_1 having this property. But \mathcal{M}_1 is a full skew monoidal subcategory of the closed \mathcal{M} therefore any type of closedness of \mathcal{M}_1 is the question of whether the corresponding internal hom functors map the unit object R into \mathcal{M}_1 . Since

$$\begin{aligned} M * _ &\cong _ \otimes_T (M \otimes_S G) \cong _ \otimes_T G \cong _ \otimes_A (T(A)A_A) \\ _ * M &\cong _ \otimes_S (M \otimes_T G) \cong _ \otimes_S G \cong _ \otimes_A (Q(A)A_A) \end{aligned}$$

for all object $M \in \mathcal{M}_1$, the right and left internal homs are

$$(97) \quad [M, N]^r \cong N_{T(A)}$$

$$(98) \quad [M, N]^\ell \cong N_{Q(A)}.$$

respectively, where, e.g. $N_{T(A)}$ denotes the set N equipped with right A -action $\langle n, a \rangle \mapsto n \cdot T(a)$. This leads to the following result.

Proposition 5.5. *For a SMM \mathcal{A} the following conditions are equivalent:*

- (i) \mathcal{A} is right closed, i.e., the $T : A \rightarrow A$ has a right adjoint in the 2-category \mathbf{Mon} .
- (ii) The right internal hom $[R, R]^r \cong A_{T(A)}$ belongs to \mathcal{M}_1 , i.e., A is rank 1 free as right A -module via T .

(iii) $F_A \equiv {}_{S(A)}F$ is rank 1 free.

Also, the following conditions are equivalent:

- (i) \mathcal{A} is left closed, i.e., the $Q : A \rightarrow A$ has a right adjoint in the 2-category \mathbf{Mon} .
- (ii) The left internal hom $[R, R]^\ell \cong A_{Q(A)}$ belongs to \mathcal{M}_1 , i.e., A is rank 1 free as right A -module via Q .
- (iii) $G_{TQ(A)}$ is a rank 1 free right A -module.

Comparing left and right closedness, especially property (iii), we see that closedness is utterly asymmetric. In a closed SMM the bialgebroid F is both source-regular and target-regular but G is not target-regular. In order to get a symmetric notion let us say that \mathcal{A} is ‘biclosed’ if it is right closed and ‘left coclosed’, i.e., require that T has a right adjoint and Q has a left adjoint. Then a little inspection shows that \mathcal{A} is ‘biclosed’ precisely when both F and G are target-regular.

So far, in this Section, we have been working within the skew monoidal category $\mathcal{M} = \mathbf{Set}_A$ associated to the bialgebroid G . There is another construction associated to a bialgebroid, the bimonad or opmonoidal monad acting on the bimodule category ${}_A\mathbf{Set}_A$. It is this latter language on which Hopf algebroids [3] and Hopf monads [1] can be defined.

Identifying ${}_A\mathbf{Set}_A$ with \mathbf{Set}_E , where $E := A^{\text{op}} \times A$, the opmonoidal monad is given by tensoring with the E -monoid G ,

$$\begin{aligned} \hat{T}X &:= X \otimes_E G \\ \hat{\mu}_X &: (x \otimes_E g) \otimes_E h \mapsto x \otimes_E gh \\ \hat{\eta}_X &: x \mapsto x \otimes_E 1_G \end{aligned}$$

and the opmonoidal structure is

$$\begin{aligned} \hat{T}_{X,Y} &: (X \otimes_A Y) \otimes_E G \rightarrow (X \otimes_E G) \otimes_A (Y \otimes_E G) \\ & \quad (x \otimes y) \otimes g \mapsto (x \otimes g^{(1)}) \otimes (y \otimes g^{(2)}) \\ \hat{T}_0 &: A \otimes_E G \rightarrow A \\ & \quad a \otimes g \mapsto \pi(T(a)g). \end{aligned}$$

The connection between the opmonoidal monad and the canonical monad $R \otimes -$ on \mathbf{Set}_A is the isomorphism of their Eilenberg-Moore categories over the forgetful functor ${}_A\mathbf{Set}_A \rightarrow \mathbf{Set}_A$. But $R \otimes -$ is lacking any (op)monoidal structure and such properties as a bialgebroid being a Hopf algebroid can only be formulated in terms of \hat{T} . The left and right fusion morphisms associated to \hat{T} are

$$(99) \quad H_{X,Y}^\ell = (\hat{T}X \otimes_A \hat{\mu}_Y) \circ \hat{T}_{X,\hat{T}Y} : \hat{T}(X \otimes_A \hat{T}Y) \rightarrow \hat{T}X \otimes_A \hat{T}Y$$

$$(100) \quad : (x \otimes_A (y \otimes_E g)) \otimes_E h \mapsto (x \otimes_E h^{(1)}) \otimes_A (y \otimes_E gh^{(2)})$$

$$(101) \quad H_{X,Y}^r = (\hat{\mu}_X \otimes_A \hat{T}Y) \circ \hat{T}_{\hat{T}X,Y} : \hat{T}(\hat{T}X \otimes_A Y) \rightarrow \hat{T}X \otimes_A \hat{T}Y$$

$$(102) \quad : ((x \otimes_E g) \otimes_A y) \otimes_E h \mapsto (x \otimes_E gh^{(1)}) \otimes_A (y \otimes_E h^{(2)})$$

and we know from [1, Theorem 3.6] that \hat{T} is left (right) Hopf, i.e., H^ℓ (resp. H^r) is invertible, precisely when $({}_A\mathbf{Set}_A)^{\hat{T}}$ is left (right) closed and the forgetful functor to ${}_A\mathbf{Set}_A$ preserves the left (right) internal hom. If we try to translate this Theorem to the skew monoidal language we again run into the problem that left and right closed structures behave differently.

Lemma 5.6. *For a skew monoidal monoid $\mathcal{A} = \langle A, T, Q, \gamma, \eta, \varepsilon \rangle$ the following conditions are equivalent:*

- (i) *G , the right A -bialgebroid associated to \mathcal{A} , is left Hopf, i.e., the opmonoidal monad $\hat{T} = _ \otimes_E G$ is a left Hopf monad,*
- (ii) *the canonical map*

$$\text{can}^G : G \otimes_{T(A)} G \rightarrow G \otimes_A G, \quad g \otimes_{T(A)} h \mapsto h^{(1)} \otimes_A gh^{(2)}$$

is a bijection,

- (iii) *γ is invertible.*
- (iv) *the canonical map*

$$\text{can}_F : F \otimes_{Q(A)} F \rightarrow F \otimes_A F, \quad f \otimes_{T(A)} f' \mapsto f_{(1)} f' \otimes_A f_{(2)}$$

is a bijection,

- (v) *F , the left A -bialgebroid associated to \mathcal{A} , is right Hopf, i.e., the opmonoidal monad $F \otimes_{A \times A^{\text{op}}} _$ is a right Hopf monad.*

Proof. (i) \Leftrightarrow (ii) is obvious from formula (99).

(ii) \Leftrightarrow (iii) Composing the canonical map with the isomorphisms

$$\begin{aligned} \tau : G \otimes_A G &\xrightarrow{\sim} G, & g_1 \otimes_A g_2 &\mapsto g_2 S(g_1 \eta) \\ \sigma : G \otimes_{T(A)} G &\xrightarrow{\sim} G, & g_1 \otimes_{T(A)} g_2 &\mapsto T(g_1 \eta) g_2 \end{aligned}$$

we obtain a map

$$c^G : G \rightarrow G, \quad g \mapsto \mu T(\mu \delta g \eta)$$

such that $\tau \circ \text{can}^G = c^G \circ \sigma$. Using the bijection $K : A \xrightarrow{\sim} G$, $a \mapsto \mu T(a)$, we can write $c^G(g) = K(\gamma K^{-1}(g))$ from which the statement follows.

By dualizing the above arguments one obtains (iii) \Leftrightarrow (iv) and (iv) \Leftrightarrow (v). \square

SMM-s with the above five properties could be called one-sided Hopf SMM although we cannot say whether left or right due to the rival conditions (i) and (v). Notice also that these one-sided Hopf properties do not require the skew monoidal structure of \mathcal{A} to be closed.

The other Hopf property, i.e., invertibility of H^r for G or invertibility of H^ℓ for F , is equivalent to the maps

$$\begin{aligned} \text{can}'^G : G \otimes_{S(A)} G &\rightarrow G \otimes_A G, & g \otimes_{S(A)} h &\mapsto gh^{(1)} \otimes_A h^{(2)} \\ \text{can}'_F : F \otimes_{S(A)} F &\rightarrow F \otimes_A F, & f \otimes_{S(A)} f' &\mapsto f_{(1)} \otimes_A f_{(2)} f' \end{aligned}$$

being bijective. Unfortunately these conditions are not equivalent to invertibility of some simple combination of the skew monoidal structure maps. However, if we assume the \mathcal{A} is left coclosed, or, just for symmetry, that it is biclosed, then there is a simple characterization in terms of a *left* skew monoidal structure. (So far, in this paper we used ‘skew monoidal’ for a right skew monoidal structure.) If Q has a left adjoint, let’s say P with unit $i : \text{id}_A \rightarrow QP$ and counit $e : PQ \rightarrow \text{id}_A$,

then a left skew monoidal structure on the monoid A can be defined by

$$\begin{aligned} a *' b &:= eP(S(b)ia), & a, b \in A \\ \gamma' &:= eP(\mu T(i)eP(\delta i)) \\ \eta' &:= eP(\eta) \\ \varepsilon' &:= \varepsilon i. \end{aligned}$$

This structure can be obtained via a contravariant embedding (similarly to Theorem 5.4) into the right skew monoidal category ${}_A\mathbf{Set}$ induced by the right A^{op} -bialgebroid G^{coop} .

Lemma 5.7. *For a biclosed skew monoidal monoid $\mathcal{A} = \langle A, T, Q, \gamma, \eta, \varepsilon \rangle$ the following conditions are equivalent:*

- (i) G , the right A -bialgebroid associated to \mathcal{A} , is right Hopf.
- (ii) can'^G is a bijection.
- (iii) γ' is invertible.

Concluding the paper we mention that the existence of non-trivial skew monoidal monoids is an open problem. It seems that the SMM structure is very sensitive to the addition of new axioms and can easily collapse into something trivial. Let us make this precise.

Definition 5.8. A skew monoidal monoid $\langle A, T, Q, \mu, \eta, \delta, \varepsilon \rangle$ is called trivial if A is commutative, $T = Q = \text{id}_A$, ε is invertible and $\mu = \varepsilon$, $\delta = \eta = \varepsilon^{-1}$.

Lemma 5.9. *For a skew monoidal monoid \mathcal{A} any one of the following conditions alone implies that \mathcal{A} is a trivial skew monoidal monoid:*

- (i) A is finite.
- (ii) A is a cancellative monoid.
- (iii) A is commutative.
- (iv) T is an automorphism.
- (v) Q is an automorphism.
- (vi) η is invertible.
- (vii) μ is invertible.
- (viii) δ is invertible.
- (ix) ε is invertible.
- (x) $\mu \bullet \mu \cong \varepsilon$ as objects of \mathcal{A}^T .

Notice that invertibility of γ does not occur among these fatal conditions, thereby granting Hopf skew monoidal monoids the chance of being non-trivial.

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